May 2

(M^n, g) asymptotically Euclidean, $R \geq 0$, integrable of order $\sigma \geq \frac{n-2}{2}$.

If solution of Ricci flow $(M^n, g(t))$, $t \in [0, \infty)$ $g(0) = g$ exists, then $m(g(0)) \geq 0$.

"$=$" iff $(M^n, g) \cong (\mathbb{R}^n, \delta_{ij})$.

Type II_b singularity ruled out.

Type III singularity $\lim_{t \to \infty} t |Rm(t)| = 0$.

$$\varepsilon > 0 \Rightarrow |Rm(t)| \leq \frac{\varepsilon}{1+t}$$

$$g(t) \xrightarrow{C^\infty} g(\infty) \text{ as } t \to \infty.$$

<u>Shi estimate</u>

$\exists\, \delta, C > 0$ s.t. $|\nabla^k Rm| \leq C t^{-1-\delta}$.

solution of heat equation $u_t = \Delta u$

compare $|Rm|$ with u.

We impose decay control: $u(0) = u_0 = r^{-2-\sigma}$, then

$$u(t) \sim r^{-2-\sigma}$$
$$\nabla u \sim r^{-3-\sigma}.$$

We use <u>Li - Yau estimate</u>: consider $f = \log u$.

$H = t(|\nabla f|^2 - 2f_t)$. Bochner formula gives

$$(\Delta - \partial_t) H \geq --$$
$$\Rightarrow H \leq C_1$$

$$\boxed{H = t\left(\frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u}\right)}$$

$$\Rightarrow \frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u} \le \frac{C_1}{t}$$ Harnack's ineq

Thm 1. $\forall\, x, y \in M$, $0 < t_1 < t_2$

$$\frac{u(y,t_2)}{u(x,t_1)} \ge \left(\frac{t_2}{t_1}\right)^{-C_1/2} \exp\left(-\frac{d_{g(t_1)}(x,y)}{2(t_2-t_1)}(1+t_2-t_1)^2\right)$$

pf. geodesic $\gamma(t) : [t_1, t_2] \to M$ w.r.t. $g(t_1)$ s.t.

$$\begin{cases} |\dot\gamma(t)| = \dfrac{d_{g(t_1)}(x,y)}{t_2-t_1} \\ \gamma(t_1) = x, \quad \gamma(t_2) = y \end{cases}$$

$$\Rightarrow \log\frac{u(y,t_2)}{u(x,t_1)} = \int_{t_1}^{t_2} \frac{d}{dt}\left(\log u(\gamma(t))\right) dt$$

$$= \int_{t_1}^{t_2} \Big(\underbrace{\frac{\partial}{\partial t}\log u}_{\frac{u_t}{u}} + \underbrace{\nabla \log u}_{\frac{\nabla u}{u}} \frac{\partial \gamma}{\partial t}\Big) dt.$$

$$\ge \int_{t_1}^{t_2} \Big(\underbrace{\frac{|\nabla \log u|^2}{2}} - \frac{C_1}{2t} + \underbrace{\nabla \log u \cdot \frac{\partial\gamma}{\partial t}}\Big) dt.$$

$$\ge -\frac{C_1}{2}\log\left(\frac{t_2}{t_1}\right) \underbrace{- \frac{1}{2}\int_{t_1}^{t_2} \left|\frac{\partial\gamma}{\partial t}\right|^2_{g(t)} dt}.$$

$$\left(\frac{1}{2}\nabla \log u + \frac{\partial\gamma}{\partial t}\right)^2 \ge 0$$ use this

$$\int_{t_1}^{t_2} \left|\frac{\partial\gamma}{\partial t}\right|^2_{g(t)} dt \le (1+t_2-t_1)^2 \int_{t_1}^{t_2} \left|\frac{\partial\gamma}{\partial t}\right|^2_{g(t_1)} dt.$$

since $|Rm(t)| \le \dfrac{\varepsilon}{1+t}$.

$$= (1+t_2-t_1)^2 \frac{(d_{g(t_1)}(x,y))^2}{t_2-t_1}.$$

Thm 2 $\exists\, \delta, C > 0$ s.t. $u(t,x) \le \dfrac{C}{(1+t)^{1+\delta}}$.

pf. Fix $p \in \left(\frac{n}{2+\sigma}, \frac{n}{2}\right)$. since $\sigma \le n-2$, $p \ge \frac{n}{2+\sigma} > 1$.

If u s.t. $\int u^p \, dv \le C_1$, then Harnack ineq implies

$$u^p(y, 2t) \ge 2^{-C_1/2} \exp\left(\frac{-p(1+t)}{2t}\right) u^p(x,t)$$

whenever $y \in B_{g(t)}\left(x, (1+t)^{\frac{1}{2}-\varepsilon}\right)$.

$$\Rightarrow \quad C_2 \ge \int_M u^p(y, 2t)\, dv_{g(2t)}$$

$$\ge \int_B u^p(y, 2t)\, dv_{g(2t)}$$

$$\ge 2^{-C_1/2} \underbrace{\exp\left(\frac{-p(1+t)}{2t}\right)}_{\ge \exp(-p) \quad \forall t \ge 1} \mathrm{Vol}_{g(2t)}(B) \cdot u^p(x,t)$$

$$\ge C_3 \cdot \mathrm{Vol}_{g(2t)}(B) \cdot u^p(x,t).$$

We want to control the volume of B.

On $K \subset M$ compact

$$\frac{d}{dt}\left(\int_K d\mu\right) = \int_K -R \, d\mu \ge \frac{-\varepsilon}{1+t} \int_K d\mu$$

$$\rightarrow \quad \mathrm{vol}_{g(t)}(V) \ge (1+t)^{-\varepsilon} \mathrm{Vol}_{g(0)}(K).$$

So $\mathrm{Vol}_{g(2t)}(B) \ge (1+2t)^{-\varepsilon} \mathrm{Vol}_{g(0)}(B)$

$\forall x, y \in M.$

$d_{g(t)}(x, y) \leq (1+t)^{\varepsilon} d_{g(0)}(x, y).$

$$\Rightarrow \mathrm{Vol}_{g(2t)}(B) \geq (1+2t)^{-\varepsilon} \mathrm{Vol}_{g(0)}\left(B_{g(0)}(x, (1+t)^{\frac{1}{2}-2\varepsilon})\right)$$

AE condition.

$$\geq (1+2t)^{-\varepsilon} (1+t)^{(\frac{1}{2}-2\varepsilon)n} u^p(x, t).$$

Choose ε small enough s.t.

$$\left(\varepsilon - (\tfrac{1}{2} - 2\varepsilon)n\right)/p < -1.$$

$$\frac{d}{dt}\int u^p d\mu = \int p u^{p-1} u_t - R u^p \, d\mu$$

$$\leq \int p u^{p-1} \Delta u \, d\mu$$

$$\text{IBP} = \lim_{r\to\infty} \int_{S_r} p u^{p-1} \langle \nabla u, \nabla r \rangle d\sigma$$

$$- \lim_{r\to\infty} \int_{B_r} p(p-1) u^{p-2} |\nabla u|^2 d\mu$$

Check decay rate of the RHS. the first $= 0$

since $u^{p-1}\langle \nabla u, \nabla r\rangle$ has decay rate $=$

$(-2-\sigma)(p-1) - 3 - \sigma \leq -(n+1).$

$\forall T$ on M, a tensor

$$\left(\frac{\partial}{\partial t} - \Delta\right) \underbrace{\left(\frac{|T|^2}{u^2}\right)}_{=: W \text{ when } T = Rm} = \frac{2}{u} \nabla u \cdot \nabla \frac{|T|^2}{u^2} - 2 \frac{|u \cdot \nabla T - \nabla u \, T|^2}{u^4} + \frac{(\partial_t - \Delta)|T|^2}{u^2}$$

$$\rightarrow (\partial_t - \Delta) W = \frac{2}{u} \cdot \nabla u \cdot \nabla W - 2 \frac{|u \nabla Rm - \nabla u \nabla Rm|^2}{u^4} + p$$

where $p = 8(B_{ijkl} + B_{ikjl}) R_{ijkl}$

$B_{ijkl} = R_{pijq} R_{qlkp}$.

estimate for p: $p \le \frac{16\varepsilon}{1+t} W$.

By maximal principle. $W = \frac{|Rm|^2}{u^2} \le C(1+t)^{16\varepsilon}$.

which gives estimate for $|Rm|^2 \le C(1+t)^{-1-\delta}$.

higher estimate for $|\nabla^k Rm| \le C_k t^{-1-\delta-\frac{k}{2}}$ is by induction.